Discrete linear multiple recurrence with multi-periodic coefficients

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Abstract

The aim of our paper is to formulate and solve problems concerning linear multiple periodic recurrence equations. Among other things, we discuss in detail the cases with periodic and multi-periodic coefficients, highlighting in particular the theorems of Floquet type. For this aim, we find specific forms for the fundamental matrix. Explicitly monodromy matrix is given, and its eigenvalues (called Floquet multipliers) are shown. The Floquet point of view brings about an important simplification: the initial linear multiple recurrence system is reduced to another linear multiple recurrence system, with constant coefficients along partial directions. The results are applied to the discrete multitime Samuelson-Hicks models with constant, respectively multi-periodic, coefficients, in order to find bivariate sequences with economic meaning.

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1 Discrete multitime multiple recurrence

A multivariate recurrence relation is an equation that recursively defines a multivariate sequence, once one or more initial terms are given: each further term of the sequence is defined as a function of the preceding terms. Some

simply defined recurrence relations can have very complex (chaotic) behaviors, and they are a part of the field of mathematics known as nonlinear analysis. We can use such recurrences including the Differential Transform Method to solve PDEs system with initial conditions.

In this paper we shall continue the study of discrete multitime multiple recurrence, giving original results regarding Floquet theory of linear multiple periodic recurrences. As an example of application in economics, we formulate and solve a Samuelson-Hicks multitime multiplier-accelerator model. Such problems remain an area of active current research in our group. The scientific sources used by us are: general recurrence theory [1], [2], [7], [11], [21], [23], our results regarding the diagonal multitime recurrence [4], [5], Floquet theory [3], [8], [9], [10], [22] and multitime dynamical systems [12]-[20].

2 Linear multitime multiple recurrence with multi-periodic coefficients

Let $m \geq 1$ be an integer number. We denote $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbb{Z}^m$. Also, for each $\alpha \in \{1, 2, \ldots, m\}$, we denote $1_\alpha = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^m$, i.e., 1_α has 1 on the position α and 0 otherwise.

On $\mathbb{Z}^m$, we define the relation " $\leq$ ": for $t = (t^1, \ldots, t^m)$, $s = (s^1, \ldots, s^m)$,

$$s \leq t \ \text{ if } \ s^\alpha \leq t^\alpha, \ \forall \alpha \in \{1, 2, \ldots, m\}.$$

One observes that " $\leq$ " is a partial order relation on $\mathbb{Z}^m$.

Let us formulate a theory similar to those of Floquet which brings important simplification: in conditions of multiple periodicity or periodicity, a linear multitime multiple recurrence system

$$x(t + 1_\alpha) = A_\alpha(t)x(t), \quad \forall t \geq t_0, \ \forall \alpha \in \{1, 2, \ldots, m\}$$

is reduced to a linear multitime multiple recurrence system

$$y(t + 1_\alpha) = B_\alpha y(t), \quad \forall t \geq t_0, \ \forall \alpha \in \{1, 2, \ldots, m\},$$

with constant coefficients.

We denote by $\mathcal{Z}$ one of the sets $\mathbb{Z}^m$ or $\left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_1\right\}$ (with $t_1 \in \mathbb{Z}^m$).

Consider the functions $A_\alpha \colon \mathcal{Z} \to \mathcal{M}_n(K)$, $\alpha \in \{1, 2, \ldots, m\}$, which define the linear homogeneous recurrence

$$x(t + 1_\alpha) = A_\alpha(t)x(t), \quad \forall \alpha \in \{1, 2, \ldots, m\}, \tag{1}$$

with the unknown function $x \colon \left\{t \in \mathcal{Z} \,\middle|\, t \geq t_0\right\} \to K^n = \mathcal{M}_{n,1}(K)$, $t_0 \in \mathcal{Z}$.

We recall some notions and results of work [6]: Theorem 1, Propositions 1, 2, 3 and Definition 1.

Theorem 1. *a) If, for any* $(t_0, x_0) \in \mathcal{Z} \times K^n$*, there exists at least one function* $x\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to K^n$*, which, for any* $t \geq t_0$*, verifies the recurrence* (1) *and the condition* $x(t_0) = x_0$*, then*

$$A_\alpha(t + 1_\beta)A_\beta(t) = A_\beta(t + 1_\alpha)A_\alpha(t), \tag{2}$$

$$\forall t \in \mathcal{Z}, \quad \forall \alpha, \beta \in \{1, 2, \dots, m\}.$$

b) If the relations (2)*, are satisfied, then, for any* $(t_0, x_0) \in \mathcal{Z} \times K^n$*, there exists a unique function* $x\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to K^n$*, which, for any* $t \geq t_0$ *verifies the recurrence* (1) *and the condition* $x(t_0) = x_0$*.*

c) Let us suppose that $\mathcal{Z} = \mathbb{Z}^m$ *and that, for any* $\alpha \in \{1, 2, \dots, m\}$ *and any* $t \in \mathbb{Z}^m$*, the matrix* $A_\alpha(t)$ *is invertible. If the relations* (2)*, are satisfied, then, for any pair* $(t_0, x_0) \in \mathbb{Z}^m \times K^n$*, there exists a unique function* $x\colon \mathbb{Z}^m \to K^n$*, which, for any* $t \in \mathbb{Z}^m$*, verifies the relations* (1)*, and also the condition* $x(t_0) = x_0$*.*

Proposition 1. *Suppose that the relations* (2) *hold true.*

For each $t_0 \in \mathcal{Z}$ *and* $X_0 \in \mathcal{M}_n(K)$ *there exists a unique matrix solution*

$$X\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to \mathcal{M}_n(K)$$

of the recurrence

$$X(t + 1_\alpha) = A_\alpha(t)X(t), \quad \forall \alpha \in \{1, 2, \dots, m\}, \tag{3}$$

with the condition $X(t_0) = X_0$*.*

For each $t_0 \in \mathcal{Z}$, we denote

$$\chi(\,\cdot\,, t_0)\colon \{t \in \mathcal{Z} \,|\, t \geq t_0\} \to \mathcal{M}_n(K),$$

the unique matrix solution of the recurrence (3), which verifies $X(t_0) = I_n$.

Definition 1. *Suppose that the relations* (2) *hold true. The matrix function*

$$\chi(\,\cdot\,,\cdot\,)\colon \{(t, s) \in \mathcal{Z} \times \mathcal{Z} \,|\, t \geq s\} \to \mathcal{M}_n(K)$$

is called transition (fundamental) matrix associated to the linear homogeneous recurrence (1)*.*

For $\alpha \in \{1, 2, \dots, m\}$ and $k \in \mathbb{N}$, we define the function

$$C_{\alpha,k}\colon \mathcal{Z} \to \mathcal{M}_n(K),$$

$$C_{\alpha,k}(t) = \begin{cases} \displaystyle\prod_{j=1}^{k} A_\alpha(t + (k - j) \cdot 1_\alpha) & \text{if} \quad k \geq 1 \\ \qquad I_n & \text{if} \quad k = 0. \end{cases} \tag{4}$$

Proposition 2. *Suppose that the relations* (2) *hold true.*

The matrix functions $\chi(\cdot)$ *and* $C_{\alpha,k}(\cdot)$ *have the properties:*

a) $\chi(t,s)\chi(s,r)=\chi(t,r),\ \ \forall t,s,r\in\mathcal{Z}$*, with* $t\geq s\geq r$*.*

b) $\chi(s,s)=I_n,\ \ \forall s\in\mathcal{Z}$*.*

c) $\chi(t+k\cdot 1_\alpha,s)=C_{\alpha,k}(t)\cdot\chi(t,s),\ \ \forall k\in\mathbb{N},\ \forall t,s\in\mathcal{Z}$*, with* $t\geq s$*.*

d) $C_{\alpha,k}(t)=\chi(t+k\cdot 1_\alpha,t),\ \ \forall k\in\mathbb{N},\ \forall t\in\mathcal{Z}$*.*

e) $C_{\alpha,k}(t+p\cdot 1_\beta)C_{\beta,p}(t)=C_{\beta,p}(t+k\cdot 1_\alpha)C_{\alpha,k}(t),\ \ \forall k,p\in\mathbb{N},\ \forall t\in\mathcal{Z}$*.*

f) For any $t,s\in\mathcal{Z}$ *with* $t\geq s$*, we have*

$$\chi(t,s)=\prod_{\alpha=1}^{m-1}C_{\alpha,t^\alpha-s^\alpha}(s^1,...,s^\alpha,t^{\alpha+1},...,t^m)\cdot C_{m,\,t^m-s^m}(s^1,s^2,\ldots,s^{m-1},s^m).$$

g) For any $t,s\in\mathcal{Z}$*, with* $t\geq s$*, the fundamental matrix* $\chi(t,s)$ *is invertible if and only if, for any* $\alpha\in\{1,2,\ldots,m\}$ *and any* $t\in\mathcal{Z}$*, the matrix* $A_\alpha(t)$ *is invertible.*

h) For any $\alpha\in\{1,2,\ldots,m\}$*, any* $k\in\mathbb{N}$ *and for any* $t\in\mathcal{Z}$*, the matrix* $C_{\alpha,k}(t)$ *is invertible if and only if, for any* $\alpha\in\{1,2,\ldots,m\}$ *and any* $t\in\mathcal{Z}$*, the matrix* $A_\alpha(t)$ *is invertible.*

i) If $\forall\alpha\in\{1,2,\ldots,m\}$*,* $\forall t\in\mathcal{Z}$*, the matrix* $A_\alpha(t)$ *is invertible, then* $\forall t,s,t_0\in\mathcal{Z}$*, with* $t\geq s\geq t_0$*, we have* $\chi(t,s)=\chi(t,t_0)\chi(s,t_0)^{-1}$*.*

j) If $\forall\alpha\in\{1,2,\ldots,m\}$*, the matrix functions* $A_\alpha(\cdot)$ *are constant, then*

$$C_{\alpha,k}(t)=A_\alpha^k,\quad\forall k\in\mathbb{N},\forall t\in\mathbb{Z}^m,\forall\alpha\in\{1,2,\ldots,m\},$$

$$\chi(t,s)=A_1^{(t^1-s^1)}A_2^{(t^2-s^2)}\cdot\ldots\cdot A_m^{(t^m-s^m)},\quad\forall t,s\in\mathbb{Z}^m,\ \text{with}\ t\geq s.$$

Proposition 3. *We consider the functions* $A_\alpha\colon\mathcal{Z}\to\mathcal{M}_n(K)$*,* $\alpha\in\{1,\ldots,m\}$*, for which the relations* (2) *are satisfied. Let* $(t_0,x_0)\in\mathcal{Z}\times K^n$*. Then, the unique function* $x\colon\{t\in\mathcal{Z}\mid t\geq t_0\}\to K^n=\mathcal{M}_{n,1}(K)$*, which, for any* $t\geq t_0$*, verifies the recurrence* (1) *and the initial condition* $x(t_0)=x_0$*, is*

$$x(t)=\chi(t,t_0)x_0,\quad\forall t\geq t_0.$$

If $\forall\alpha\in\{1,2,\ldots,m\}$*, the matrix functions* $A_\alpha(\cdot)$ *are constants, then*

$$x(t)=A_1^{(t^1-t_0^1)}A_2^{(t^2-t_0^2)}\cdot\ldots\cdot A_m^{(t^m-t_0^m)}x_0,\quad\forall t\geq t_0.\tag{5}$$

2.1 Case of multi-periodic coefficients

Definition 2. *A function* $f\colon\mathcal{Z}\to M$ *is called multi-periodic if there exists* $(T_1,T_2,\ldots,T_m)\in\mathbb{N}^m\setminus\{0\}$ *such that*

$$f(t+T_\alpha\cdot 1_\alpha)=f(t),\quad\forall t\in\mathcal{Z},\quad\forall\alpha\in\{1,2,\ldots,m\},$$

i.e., $T_1\cdot 1_1$*,* $T_2\cdot 1_2$*, …,* $T_m\cdot 1_m$ *are periods for the function* f*.*

Proposition 4. *Let* $T = (T_1, T_2, \ldots, T_m) \in \mathbb{N}^m$*,* $T \neq 0$*.*

We consider the matrix functions $A_\alpha \colon \mathcal{Z} \to \mathcal{M}_n(K)$*,* $\alpha \in \{1, 2, \ldots, m\}$*, for which the relations* (2) *are satisfied. Suppose that,* $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$*,* $\forall t \in \mathcal{Z}$*, we have* $A_\alpha(t + T_\beta \cdot 1_\beta) = A_\alpha(t)$*. Then*

a) $C_{\alpha,k}(t + T_\beta \cdot 1_\beta) = C_{\alpha,k}(t), \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}, \ \forall t \in \mathcal{Z}$*;*

b) $C_{\alpha,T_\alpha}(t) C_{\beta,T_\beta}(t) = C_{\beta,T_\beta}(t) C_{\alpha,T_\alpha}(t), \ \forall t \in \mathcal{Z}$*;*

c) $\chi(t + T_\alpha \cdot 1_\alpha, s) = \chi(t, s) \cdot C_{\alpha,T_\alpha}(s), \ \forall t, s \in \mathcal{Z}$*, with* $t \geq s$*.*

Proof. *a*) It follows from the definition of $C_{\alpha,k}(\cdot)$; this is either the constant function I_n, or a product of multiperiodic matrix functions.

b) According the step *a*), we have

$$C_{\alpha,T_\alpha}(t + T_\beta \cdot 1_\beta) = C_{\alpha,T_\alpha}(t), \quad C_{\beta,T_\beta}(t + T_\alpha \cdot 1_\alpha) = C_{\beta,T_\beta}(t).$$

According the Proposition 2, *e*), we have

$$C_{\alpha,T_\alpha}(t + T_\beta \cdot 1_\beta) C_{\beta,T_\beta}(t) = C_{\beta,T_\beta}(t + T_\alpha \cdot 1_\alpha) C_{\alpha,T_\alpha}(t).$$

It follows that $C_{\alpha,T_\alpha}(t) C_{\beta,T_\beta}(t) = C_{\beta,T_\beta}(t) C_{\alpha,T_\alpha}(t)$.

c) Fix α. Fix $s \in \mathcal{Z}$. Let $Y_1, Y_2 \colon \{t \in \mathcal{Z} \,|\, t \geq s\} \to \mathcal{M}_n(K)$,

$$Y_1(t) = \chi(t + T_\alpha \cdot 1_\alpha, s), \quad Y_2(t) = \chi(t, s) \cdot C_{\alpha,T_\alpha}(s), \quad \forall t \geq s.$$

For each $\beta \in \{1, 2, \ldots, m\}$, we have

$$Y_1(t + 1_\beta) = \chi(t + T_\alpha \cdot 1_\alpha + 1_\beta, s) = A_\beta(t + T_\alpha \cdot 1_\alpha) \chi(t + T_\alpha \cdot 1_\alpha, s) = A_\beta(t) Y_1(t);$$

$$Y_2(t + 1_\beta) = \chi(t + 1_\beta, s) \cdot C_{\alpha,T_\alpha}(s) = A_\beta(t) \chi(t, s) \cdot C_{\alpha,T_\alpha}(s) = A_\beta(t) Y_2(t).$$

Consequently the functions $Y_1(\cdot)$ and $Y_2(\cdot)$ are both solutions of the recurrence (3).

According the Proposition 2, *d*), we have $\chi(s + T_\alpha \cdot 1_\alpha, s) = C_{\alpha,T_\alpha}(s)$, which is equivalent to $\chi(s + T_\alpha \cdot 1_\alpha, s) = \chi(s, s) C_{\alpha,T_\alpha}(s)$, i.e., $Y_1(s) = Y_2(s)$. From the uniqueness property (Proposition 1) it follows that $Y_1(\cdot)$ and $Y_2(\cdot)$ coincide; hence $\chi(t + T_\alpha \cdot 1_\alpha, s) = \chi(t, s) \cdot C_{\alpha,T_\alpha}(s)$, $\forall t \geq s$. $\square$

It proves without difficulty the following result.

Lemma 1. *Let* $A, B \in \mathcal{M}_2(\mathbb{C})$*, such that* $A \neq \lambda I_2$*,* $\forall \lambda \in \mathbb{C}$*. Then*

$AB = BA$ *if and only if there exist* $z, w \in \mathbb{C}$*, such that* $B = zI_2 + wA$*.*

Proposition 5. *Suppose that the matrices $P_1, P_2, \ldots, P_m \in \mathcal{M}_n(\mathbb{C})$ commute, i.e., $P_\alpha P_\beta = P_\beta P_\alpha$, $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$.*

Let us assume that one of the statements i) or ii) below is true.

i) For each $\alpha \in \{1, 2, \ldots, m\}$, the matrix P_α is diagonalizable.

ii) $n = 2$ and for any $\alpha \in \{1, 2, \ldots, m\}$, the matrix P_α is invertible.

Let $k_1, k_2, \ldots, k_m \in \mathbb{N}^*$. Then, for any $\alpha \in \{1, 2, \ldots, m\}$, there exists $Q_\alpha \in \mathcal{M}_n(\mathbb{C})$ such that

a) $Q_\alpha^{k_\alpha} = P_\alpha$, $\forall \alpha \in \{1, 2, \ldots, m\}$,

b) $Q_\alpha Q_\beta = Q_\beta Q_\alpha$, $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$.

Proof. For $d_1, d_2, \ldots, d_n \in \mathbb{C}$, we denote $\operatorname{diag}(d_1, d_2, \ldots, d_n)$, the matrix from $\mathcal{M}_n(\mathbb{C})$, with main diagonal elements $d_1, d_2, \ldots, d_n$ (in this order) and in rest 0.

If Hypothesis *i*) is true and if $P_\alpha P_\beta = P_\beta P_\alpha$, $\forall \alpha, \beta$, then the matrices P_α are simultaneously diagonalizable, i.e. there exist $T \in \mathcal{M}_n(\mathbb{C})$, T invertible and there exist $\lambda_{\alpha,1}$, $\lambda_{\alpha,2}$, $\ldots$, $\lambda_{\alpha,n} \in \mathbb{C}$ such that

$$P_\alpha = T \cdot \operatorname{diag}(\lambda_{\alpha,1}, \lambda_{\alpha,2}, \ldots, \lambda_{\alpha,n}) \cdot T^{-1}, \quad \forall \alpha \in \{1, 2, \ldots, m\}.$$

There exist $\theta_{\alpha,j} \in \mathbb{C}$, such that $\theta_{\alpha,j}^{k_\alpha} = \lambda_{\alpha,j}$, $\forall \alpha \in \{1, 2, \ldots, m\}$, $\forall j \in \{1, 2, \ldots, n\}$.

For $Q_\alpha = T \cdot \operatorname{diag}(\theta_{\alpha,1}, \theta_{\alpha,2}, \ldots, \theta_{\alpha,n}) \cdot T^{-1}$, $\alpha \in \{1, 2, \ldots, m\}$, easily finds that statements *a*) and *b*) hold.

Suppose that the hypothesis *ii*) is true.

If $\forall \alpha$, $\exists \lambda_\alpha \in \mathbb{C}$, such that $P_\alpha = \lambda_\alpha I_2$, then the matrices P_α are diagonalizable and hence the hypothesis *i*) is true, and this case has already been treated.

We left to study the case in which at least one of the matrices P_α is not of the form $P_\alpha = \lambda_\alpha I_2$, with $\lambda_\alpha \in \mathbb{C}$. After a possible renumbering we can assume that this matrix is P_1. Hence $P_1 \neq \lambda I_2$, $\forall \lambda \in \mathbb{C}$.

Since $P_1 P_\alpha = P_\alpha P_1$, from the Lemma 1 follows that there exist $z_\alpha, w_\alpha \in \mathbb{C}$, such that $P_\alpha = z_\alpha I_2 + w_\alpha P_1$ ($\forall \alpha \in \{1, 2, \ldots, m\}$).

If P_1 is diagonalizable, then the matrix $z_\alpha I_2 + w_\alpha P_1$ is also diagonalizable, So assuming *i*) is satisfied, and this case has already been treated.

If the matrix P_1 is not diagonalizable, then there exists a matrix $T \in \mathcal{M}_2(\mathbb{C})$, T invertible and there exists $\lambda \in \mathbb{C}$, such that $P_1 = T \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} T^{-1}$.

Consequently, we have $P_1 = \lambda I_2 + S$, where $S = T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} T^{-1}$.

We observe that $P_\alpha = (z_\alpha + \lambda w_\alpha) I_2 + w_\alpha S$, $\forall \alpha \in \{1, 2, \ldots, m\}$. Since $S^2 = O_2$, we deduce that the matrix S is not invertible; hence $z_\alpha + \lambda w_\alpha \neq 0$.

It follows that there exists $u_\alpha \in \mathbb{C}$, $u_\alpha \neq 0$, such that $u_\alpha^{k_\alpha} = z_\alpha + \lambda w_\alpha$. Let $v_\alpha = \dfrac{w_\alpha}{k_\alpha u_\alpha^{k_\alpha - 1}}$.

We choose the matrix $Q_\alpha = u_\alpha I_2 + v_\alpha S$. Since $S^j = O_2$, $\forall j \geq 2$, we obtain

$$Q_\alpha^{k_\alpha} = u_\alpha^{k_\alpha} I_2 + k_\alpha u_\alpha^{k_\alpha - 1} v_\alpha S = (z_\alpha + \lambda w_\alpha) I_2 + w_\alpha S = P_\alpha,\ \forall \alpha \in \{1, \ldots, m\}.$$

The equality $Q_\alpha Q_\beta = Q_\beta Q_\alpha$ is obvious. $\square$

Proposition 6. *Let $t_0 \in \mathbb{Z}^m$, fixed. We denote $\mathcal{Z} := \big\{t \in \mathbb{Z}^m \,\big|\, t \geq t_0\big\}$.*

Let $T = (T_1, T_2, \ldots, T_m) \in \mathbb{N}^m$, $T \neq 0$.

Consider the matrix functions $A_\alpha \colon \mathcal{Z} \to \mathcal{M}_n(\mathbb{C})$, $\alpha \in \{1, 2, \ldots, m\}$, for which the relations (2) *are satisfied. Suppose that, $\forall \alpha, \beta \in \{1, 2, \ldots, m\}$, $\forall t \in \mathcal{Z}$, $A_\alpha(t + T_\beta \cdot 1_\beta) = A_\alpha(t)$ and $A_\alpha(t)$ is invertible.*

We denote $\Phi(t) := \chi(t, t_0)$, $t \in \mathcal{Z}$, and $\widetilde{C}_\alpha = C_{\alpha, T_\alpha}(t_0)$.

Let $F_1 = \big\{\alpha \in \{1, \ldots, m\} \,\big|\, T_\alpha \geq 1\big\}$, $F_2 = \big\{\alpha \in \{1, \ldots, m\} \,\big|\, T_\alpha = 0\big\}$.

For each $\alpha \in F_1$, we choose $B_\alpha \in \mathcal{M}_n(\mathbb{C})$ such that $B_\alpha^{T_\alpha} = \widetilde{C}_\alpha$.

For each $\alpha \in F_2$, we choose $B_\alpha = I_n$.

If for any $\alpha, \beta \in F_1$, we have $B_\alpha B_\beta = B_\beta B_\alpha$, then there exists a function, $P \colon \mathcal{Z} \to \mathcal{M}_n(\mathbb{C})$, such that

$P(t + T_\alpha \cdot 1_\alpha) = P(t)$, $\quad \forall t \in \mathcal{Z}$, $\forall \alpha \in \{1, 2, \ldots, m\}$, and

$\Phi(t) = P(t) B_1^{t^1} B_2^{t^2} \cdot \ldots \cdot B_m^{t^m}$, $\quad \forall t \geq t_0$.

Proof. We remark that for any $\alpha, \beta \in \{1, 2, \ldots, m\}$, we have $B_\alpha B_\beta = B_\beta B_\alpha$.

If $\alpha \in F_2$, i.e. $T_\alpha = 0$, then $\widetilde{C}_\alpha = I_n$. We observe that the equality $B_\alpha^{T_\alpha} = \widetilde{C}_\alpha$ is true. Hence, for any $\alpha \in \{1, 2, \ldots, m\}$, we have $\widetilde{C}_\alpha = B_\alpha^{T_\alpha}$.

Let $\alpha \in F_1$, i.e. $T_\alpha \geq 1$. Since the matrix $\widetilde{C}_\alpha$ is invertible (Proposition 2), from $B_\alpha^{T_\alpha} = \widetilde{C}_\alpha$ it follows that the matrix B_α is invertible.

Hence, for any $\alpha \in \{1, 2, \ldots, m\}$, the matrix B_α is invertible.

We define the function $P \colon \mathcal{Z} \to \mathcal{M}_n(\mathbb{C})$,

$$P(t) = \Phi(t) B_1^{-t^1} B_2^{-t^2} \cdot \ldots \cdot B_m^{-t^m}, \quad \forall t \geq t_0. \tag{6}$$

Since from the formula (6) it follows immediately the equality

$$\Phi(t) = P(t) B_1^{t^1} B_2^{t^2} \cdot \ldots \cdot B_m^{t^m},$$

it is sufficient to show also that $P(\cdot)$ is a multiperiodic function:

$$P(t + T_\alpha \cdot 1_\alpha) = \chi(t + T_\alpha \cdot 1_\alpha, t_0) B_1^{-t^1} B_2^{-t^2} \cdot \ldots \cdot B_\alpha^{-t^\alpha - T_\alpha} \cdot \ldots \cdot B_m^{-t^m}.$$

According the Proposition 4, c), we have

$$\chi(t+T_\alpha\cdot 1_\alpha,t_0)=\chi(t,t_0)\cdot C_{\alpha,T_\alpha}(t_0)=\Phi(t)\widetilde{C}_\alpha=\Phi(t)B_\alpha^{T_\alpha}.$$

We obtain

$$P(t+T_\alpha\cdot 1_\alpha)=\Phi(t)B_\alpha^{T_\alpha}B_1^{-t^1}B_2^{-t^2}\cdot\ldots\cdot B_\alpha^{-t^\alpha-T_\alpha}\cdot\ldots\cdot B_m^{-t^m}=$$

$$=\Phi(t)B_1^{-t^1}B_2^{-t^2}\cdot\ldots\cdot B_\alpha^{-t^\alpha}\cdot\ldots\cdot B_m^{-t^m}=P(t).$$

□

Theorem 2. *Let $t_0\in\mathbb{Z}^m$, fixed. We denote $\mathcal{Z}:=\{t\in\mathbb{Z}^m\,|\,t\geq t_0\}$.*

Let $T=(T_1,T_2,\ldots,T_m)\in\mathbb{N}^m$, $T\neq 0$.

Consider the matrix functions $A_\alpha\colon\mathcal{Z}\to\mathcal{M}_n(\mathbb{C})$, $\alpha\in\{1,2,\ldots,m\}$, for which the relations (2) *are satisfied. Suppose that, $\forall\alpha,\beta\in\{1,2,\ldots,m\}$, $\forall t\in\mathcal{Z}$, $A_\alpha(t+T_\beta\cdot 1_\beta)=A_\alpha(t)$ and $A_\alpha(t)$ is invertible.*

We denote $\Phi(t):=\chi(t,t_0)$, $t\in\mathcal{Z}$.

Let us assume that one of the statements i) and ii) below is true.

i) $n=2$.

ii) $\forall\alpha\in\{1,2,\ldots,m\}$, $\forall t\in\mathcal{Z}$, the matrix $A_\alpha(t)$ is Hermitian and

$$A_\alpha(t)A_\alpha(t+k\cdot 1_\alpha)=A_\alpha(t+k\cdot 1_\alpha)A_\alpha(t),\ \forall t\in\mathcal{Z},\ \forall k\in\mathbb{N},\ \forall\alpha\in\{1,\ldots,m\}.$$

Then there exists a function, $P\colon\mathcal{Z}\to\mathcal{M}_n(\mathbb{C})$, and also there exist the constant invertible matrices B_1, B_2, …, $B_m\in\mathcal{M}_n(\mathbb{C})$, such that

a) $P(t+T_\alpha\cdot 1_\alpha)=P(t)$, $\quad\forall t\in\mathcal{Z}$, $\forall\alpha\in\{1,2,\ldots,m\}$;

b) $B_\alpha B_\beta=B_\beta B_\alpha$, $\quad\forall\alpha,\beta\in\{1,2,\ldots,m\}$;

c) $\Phi(t)=P(t)B_1^{t^1}B_2^{t^2}\cdot\ldots\cdot B_m^{t^m}$, $\quad\forall t\geq t_0$.

Proof. Let $\widetilde{C}_\alpha=C_{\alpha,T_\alpha}(t_0)$. The matrices $\widetilde{C}_\alpha$ are invertible (Proposition 2).

One observes that in the hypothesis ii), the matrices $\widetilde{C}_\alpha$ are Hermitian, hence diagonalizable.

Let $F_1=\{\alpha\in\{1,\ldots,m\}\,|\,T_\alpha\geq 1\}$, $F_2=\{\alpha\in\{1,\ldots,m\}\,|\,T_\alpha=0\}$.

For each $\alpha\in F_2$, i.e. $T_\alpha=0$, we choose $B_\alpha=I_n$.

For the set of invertible matrices $\{\widetilde{C}_\alpha\,|\,\alpha\in F_1\}$, we apply the Proposition 5. The Proposition 4 says that $\widetilde{C}_\alpha\widetilde{C}_\beta=\widetilde{C}_\beta\widetilde{C}_\alpha$. Note that either $n=2$ or the matrices $\widetilde{C}_\alpha$ are diagonalizable. Hence the hypotheses of the Proposition 5 are true. Hence, for each $\alpha\in F_1$, there exists $B_\alpha\in\mathcal{M}_n(\mathbb{C})$ such that $B_\alpha^{T_\alpha}=\widetilde{C}_\alpha$ and

$$B_\alpha B_\beta=B_\beta B_\alpha,\quad\forall\alpha,\beta\in F_1.$$

From $B_\alpha^{T_\alpha}=\widetilde{C}_\alpha$ and $T_\alpha\geq 1$, it follows that the matrix T_α is invertible.

Since for $\alpha \in F_2$, we have $B_\alpha = I_n$, it follows

$$B_\alpha B_\beta = B_\beta B_\alpha, \quad \forall \alpha, \beta \in \{1, 2, \ldots, m\}.$$

We observe that the hypotheses of the Proposition 6 are true. Consequently, it follows automatically the points $a)$ and $c)$. $\square$

Theorem 3. *Let $t_0 \in \mathbb{Z}^m$, fixed. We denote $\mathcal{Z} := \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_0\right\}$.*

Let $T = (T_1, T_2, \ldots, T_m) \in \mathbb{N}^m$, $T \neq 0$.

Consider the matrix functions $A_\alpha \colon \mathcal{Z} \to \mathcal{M}_n(\mathbb{C})$, $\alpha \in \{1, 2, \ldots, m\}$, for which the relations (2) *are satisfied. Suppose that, $\forall \alpha \in \{1, 2, \ldots, m\}$, $A_\alpha(t)$ is invertible, $\forall t \in \mathcal{Z}$. We denote $\Phi(t) := \chi(t, t_0)$, $t \in \mathcal{Z}$.*

We assume that there exists a function $P \colon \mathcal{Z} \to \mathcal{M}_n(\mathbb{C})$ and there exist the constant invertible matrices B_1, B_2, $\ldots$, $B_m \in \mathcal{M}_n(\mathbb{C})$, such that the relations from the steps a), b), c) of Theorem 2, to be satisfied.

We consider the recurrences

$$x(t + 1_\alpha) = A_\alpha(t)x(t), \quad \forall t \geq t_0, \ \forall \alpha \in \{1, 2, \ldots, m\}; \tag{7}$$

$$y(t + 1_\alpha) = B_\alpha y(t), \quad \forall t \geq t_0, \ \forall \alpha \in \{1, 2, \ldots, m\}. \tag{8}$$

If $y(t)$ is a solution of the recurrence (8)*, then $x(t) := P(t)y(t)$ is a solution of the recurrence* (7)*. And conversely, if $x(t)$ is a solution of the recurrence* (7)*, then $y(t) := P(t)^{-1}x(t)$ is a solution of the recurrence* (8)*.*

Proof. Since the matrices B_α commute, it follows that the recurrence (8) has the existence and uniqueness property of solutions (see Theorem 1).

The matrix $\Phi(t)$ is invertible (Proposition 2). From the equality of the point $b)$ in the Theorem 2, it follows that the matrix $P(t)$ is invertible.

Let $y(t)$ be a solution of the recurrence (8) and $x(t) := P(t)y(t)$; hence $y(t) := P(t)^{-1}x(t)$.

$$y(t + 1_\alpha) = B_\alpha y(t) \Longleftrightarrow P(t + 1_\alpha)^{-1}x(t + 1_\alpha) = B_\alpha P(t)^{-1}x(t)$$

$$\Longleftrightarrow x(t + 1_\alpha) = P(t + 1_\alpha)B_\alpha P(t)^{-1}x(t).$$

$$\Longleftrightarrow x(t + 1_\alpha) = \Phi(t + 1_\alpha)B_1^{-t^1}B_2^{-t^2} \cdot \ldots \cdot B_\alpha^{-t^\alpha - 1} \cdot \ldots \cdot B_m^{-t^m} \cdot B_\alpha P(t)^{-1}x(t)$$

$$\Longleftrightarrow x(t + 1_\alpha) = A_\alpha(t)\Phi(t)B_1^{-t^1}B_2^{-t^2} \cdot \ldots \cdot B_\alpha^{-t^\alpha} \cdot \ldots \cdot B_m^{-t^m} \cdot P(t)^{-1}x(t)$$

$$\Longleftrightarrow x(t + 1_\alpha) = A_\alpha(t)P(t) \cdot P(t)^{-1}x(t) \Longleftrightarrow x(t + 1_\alpha) = A_\alpha(t)x(t).$$

Like it proves the converse. $\square$

Conjecture 1. *Suppose that the invertible matrices $P_1, P_2, \dots, P_m \in \mathcal{M}_n(\mathbb{C})$ commute, i.e., $P_\alpha P_\beta = P_\beta P_\alpha$, $\forall \alpha, \beta \in \{1, 2, \dots, m\}$.*

Let $k_1, k_2, \dots, k_m \in \mathbb{N}^*$. Then, for any $\alpha \in \{1, 2, \dots, m\}$, there exists $Q_\alpha \in \mathcal{M}_n(\mathbb{C})$ such that

a) $Q_\alpha^{k_\alpha} = P_\alpha$, $\forall \alpha \in \{1, 2, \dots, m\}$,

b) $Q_\alpha Q_\beta = Q_\beta Q_\alpha$, $\forall \alpha, \beta \in \{1, 2, \dots, m\}$.

Conjecture 2. *Let $t_0 \in \mathbb{Z}^m$, fixed. We denote $\mathcal{Z} := \{ t \in \mathbb{Z}^m \,|\, t \geq t_0 \}$.*

Let $T = (T_1, T_2, \dots, T_m) \in \mathbb{N}^m$, $T \neq 0$.

Consider the matrix functions $A_\alpha \colon \mathcal{Z} \to \mathcal{M}_n(\mathbb{C})$, $\alpha \in \{1, 2, \dots, m\}$, for which the relations (2) *are satisfied. Suppose that, $\forall \alpha, \beta \in \{1, 2, \dots, m\}$, $\forall t \in \mathcal{Z}$, $A_\alpha(t + T_\beta \cdot 1_\beta) = A_\alpha(t)$ and $A_\alpha(t)$ is invertible.*

We denote $\Phi(t) := \chi(t, t_0)$, $t \in \mathcal{Z}$.

Then there exists a function, $P \colon \mathcal{Z} \to \mathcal{M}_n(\mathbb{C})$, and also there exist the constant invertible matrices B_1, B_2, $\dots$, $B_m \in \mathcal{M}_n(\mathbb{C})$, such that

a) $P(t + T_\alpha \cdot 1_\alpha) = P(t), \quad \forall t \in \mathcal{Z}, \ \forall \alpha \in \{1, 2, \dots, m\}$;

b) $B_\alpha B_\beta = B_\beta B_\alpha, \quad \forall \alpha, \beta \in \{1, 2, \dots, m\}$;

c) $\Phi(t) = P(t) B_1^{t^1} B_2^{t^2} \cdot \ldots \cdot B_m^{t^m}, \quad \forall t \geq t_0$.

Remark 1. *From the Conjecture* 1 *it follows the Conjecture* 2.

The proof is the same as that of Theorem 2. *One uses the Conjecture* 1 *instead of the Proposition* 5 *to show that the hypotheses of the Proposition* 6 *are satisfied. Then one applies Proposition* 6.

2.2 Example and commentaries

Example 1. *Let us consider $m = 2$, $n \geq 2$, $T_2 \in \mathbb{Z}$, $T_2 \geq 2$,*

$$Q_1 = \begin{pmatrix} \cos \dfrac{\pi}{T_2} & -\sin \dfrac{\pi}{T_2} \\ \sin \dfrac{\pi}{T_2} & \cos \dfrac{\pi}{T_2} \end{pmatrix}, \quad S_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$Q = \begin{pmatrix} Q_1 & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix}, \quad S = \begin{pmatrix} S_1 & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix}, \quad \textit{dacă } n \geq 3.$$

If $n = 2$, we set $Q = Q_1$ and $S = S_1$.

We have $Q_1^{-1} = Q_1^\top$, $Q^{-1} = Q^\top$, $S_1^{-1} = S_1 = S_1^\top$, $S^{-1} = S = S^\top$, and hence the matrices Q_1, Q are orthogonal and S_1, S are symmetric and orthogonal.

Powers: $Q_1^{T_2} = -I_2$, $Q_1^{2T_2} = I_2$, $Q^{T_2} = \begin{pmatrix} -I_2 & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix}$, $Q^{2T_2} = I_n$.

Also, we have $S_1Q_1 = Q_1^{-1}S_1$ and $SQ = Q^{-1}S$. Hence $Q^{-1} = SQS$ and $Q^{-k} = SQ^kS$ or $SQ^{-k} = Q^kS$, $\forall k \in \mathbb{Z}$.

If $SQ = QS$, then $Q^{-1} = Q \Longrightarrow Q^2 = I_n \Longrightarrow \cos\dfrac{2\pi}{T_2} = 1$ and $\sin\dfrac{2\pi}{T_2} = 0$, what can not because $T_2 \geq 2$. Hence $SQ \neq QS$.

Let us consider the matrices

$$A_1, A_2 \colon \mathbb{Z}^2 \to \mathcal{M}_n(\mathbb{C}),$$

$$A_1(t^1,t^2) = Q^{t^2}SQ^{-t^2} = Q^{2t^2}S, \quad A_2(t^1,t^2) = Q, \quad \forall (t^1,t^2) \in \mathbb{Z}^2.$$

Since the matrix S is symmetric and the matrix Q is orthogonal it follows that the matrix $A_1(t^1,t^2)$ is hermitian (it's really real and symmetric). The matrix $A_1(t^1,t^2)$ is also orthogonal.

Unfortunately the matrix $A_2(t^1,t^2)$ is not hermitian (symmetric). This is the only hypothesis of Theorem 2 which does not take.

Let us compute A_1:

$$Q^{2t^2}S = \begin{pmatrix} Q_1^{2t^2}S_1 & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix}, \quad Q_1^{2t^2} = \begin{pmatrix} \cos\dfrac{2\pi t^2}{T_2} & -\sin\dfrac{2\pi t^2}{T_2} \\ \sin\dfrac{2\pi t^2}{T_2} & \cos\dfrac{2\pi t^2}{T_2} \end{pmatrix};$$

$$A_1(t^1,t^2) = \begin{pmatrix} Q_1^{2t^2}S_1 & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix}, \text{ with } Q_1^{2t^2}S_1 = \begin{pmatrix} \cos\dfrac{2\pi t^2}{T_2} & \sin\dfrac{2\pi t^2}{T_2} \\ \sin\dfrac{2\pi t^2}{T_2} & -\cos\dfrac{2\pi t^2}{T_2} \end{pmatrix};$$

$$A_1(t^1,t^2+1)A_2(t^1,t^2) = Q^{t^2+1}SQ^{-t^2-1}Q = Q^{t^2+1}SQ^{-t^2}$$
$$A_2(t^1+1,t^2)A_1(t^1,t^2) = QQ^{t^2}SQ^{-t^2} = Q^{t^2+1}SQ^{-t^2}.$$

Hence $A_1(t^1,t^2+1)A_2(t^1,t^2) = A_2(t^1+1,t^2)A_1(t^1,t^2)$, i.e., the relation (2) is true.

Obviously that $A_1(t^1,t^2)A_1(t^1+k,t^2) = A_1(t^1+k,t^2)A_1(t^1,t^2)$, $\forall k \in \mathbb{N}$, $\forall (t^1,t^2) \in \mathbb{Z}^2$, since $A_1(\cdot,\cdot)$ is constant with respect to the first argument. We have also $A_2(t^1,t^2)A_2(t^1,t^2+k) = A_2(t^1,t^2+k)A_2(t^1,t^2)$, $\forall k \in \mathbb{N}$, $\forall (t^1,t^2) \in \mathbb{Z}^2$.

Now, obviously that $A_1(t^1+1,t^2) = A_1(t^1,t^2)$, $\forall (t^1,t^2) \in \mathbb{Z}^2$. $A_1(t^1,t^2+T_2) = Q^{2t^2+2T_2}S = Q^{2t^2}S$, since $Q^{2T_2} = I_n$. Hence $A_1(t^1,t^2+T_2) = A_1(t^1,t^2)$, $\forall (t^1,t^2) \in \mathbb{Z}^2$.

Since $A_2(\cdot,\cdot)$ is a constant function, it follows that

$$A_2(t^1+1,t^2) = A_2(t^1,t^2), \; A_2(t^1,t^2+T_2) = A_2(t^1,t^2), \; \forall (t^1,t^2) \in \mathbb{Z}^2.$$

Was also observed that $A_1(t^1,t^2)A_2(t^1,t^2) \neq A_2(t^1,t^2)A_1(t^1,t^2)$, $\forall (t^1,t^2)$, because if we had equality would result $Q^{t^2}SQ^{-t^2}Q = QQ^{t^2}SQ^{-t^2}$, equivalent to $SQ = QS$, what is false (we proved above that $SQ \neq QS$).

According Theorem 1, for any $t_0 = (t_0^1, t_0^2) \in \mathbb{Z}^2$ and any $x_0 \in \mathbb{C}^n$, there exists a unique solution $x\colon \mathbb{Z}^2 \to \mathbb{C}^n$ of the double recurrence

$$\begin{cases} x(t^1+1,t^2) = A_1(t^1,t^2)x(t^1,t^2) \\ x(t^1,t^2+1) = A_2(t^1,t^2)x(t^1,t^2), \end{cases} \quad \forall (t^1,t^2) \in \mathbb{Z}^2, \tag{9}$$

which verifies the initial condition $x(t_0) = x_0$. Let us show that this solution can be written in the form

$$x(t^1,t^2) = Q^{t^2}S^{t^1-t_0^1}Q^{-t_0^2}x_0, \quad \forall (t^1,t^2) \in \mathbb{Z}^2. \tag{10}$$

Indeed,

$$x(t^1+1,t^2) = Q^{t^2}S^{t^1+1-t_0^1}Q^{-t_0^2}x_0,$$

$$A_1(t^1,t^2)x(t^1,t^2) = Q^{t^2}SQ^{-t^2}Q^{t^2}S^{t^1-t_0^1}Q^{-t_0^2}x_0 = Q^{t^2}S^{1+t^1-t_0^1}Q^{-t_0^2}x_0.$$

$$x(t^1,t^2+1) = Q^{t^2+1}S^{t^1-t_0^1}Q^{-t_0^2}x_0 = QQ^{t^2}S^{t^1-t_0^1}Q^{-t_0^2}x_0 = A_2(t^1,t^2)x(t^1,t^2)$$

and the equality $x(t_0^1,t_0^2) = x_0$ is obvious.

It follows that the fundamental matrix is $\chi(t,t_0) = Q^{t^2}S^{t^1-t_0^1}Q^{-t_0^2}$. We select $t_0 = (0,0)$; let $\Phi(t^1,t^2) := \chi\big((t^1,t^2);(0,0)\big) = Q^{t^2}S^{t^1}$.

We shall determine the matrices B_1, B_2 and $P(\cdot)$ as in Proposition 6, with $T_1 = 1$.

$$\widetilde{C}_1 = C_{1,T_1}(t_0) = C_{1,1}(0,0) = A_1(0,0) = S,$$

$$\widetilde{C}_2 = C_{2,T_2}(0,0) = Q^{T_2} = \begin{pmatrix} -I_2 & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix}.$$

We look for the matrices $B_1, B_2 \in \mathcal{M}_n(\mathbb{C})$ that satisfy $B_1^{T_1} = \widetilde{C}_1$, $B_2^{T_2} = \widetilde{C}_2$, i.e., $B_1 = S$, and $B_2^{T_2} = \begin{pmatrix} -I_2 & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix}$. Since there exists a complex number $z \in \mathbb{C}$ such that $z^{T_2} = -1$, we select $B_2 = \begin{pmatrix} zI_2 & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix}$ and conclude that $B_2^{T_2} = \begin{pmatrix} -I_2 & O_{2,n-2} \\ O_{n-2,2} & I_{n-2} \end{pmatrix}$.

We have also $B_1B_2 = B_2B_1$ (i.e., $SB_2 = B_2S$), since B_1 and B_2 are diagonal matrices. We see that the assumptions of Proposition 6 are satisfied (with $T_1 = 1$).

The relation $\Phi(t^1,t^2) = P(t^1,t^2)B_1^{t^1}B_2^{t^2}$ is equivalent to

$$Q^{t^2}S^{t^1} = P(t^1,t^2)B_1^{t^1}B_2^{t^2} \iff Q^{t^2}S^{t^1} = P(t^1,t^2)B_2^{t^2}S^{t^1} \iff P(t^1,t^2) = Q^{t^2}B_2^{-t^2}$$

$$P(t^1,t^2)=\begin{pmatrix} Q_1^{t^2} & O_{2,n-2}\\ O_{n-2,2} & I_{n-2}\end{pmatrix}\begin{pmatrix} z^{-t^2}I_2 & O_{2,n-2}\\ O_{n-2,2} & I_{n-2}\end{pmatrix}=\begin{pmatrix} z^{-t^2}Q_1^{t^2} & O_{2,n-2}\\ O_{n-2,2} & I_{n-2}\end{pmatrix}$$

$$P(t^1+T_1,t^2)=P(t^1+1,t^2)=Q^{t^2}B_2^{-t^2}=P(t^1,t^2)$$

$$P(t^1,t^2+T_2)=\begin{pmatrix} z^{-t^2-T_2}Q_1^{t^2+T_2} & O_{2,n-2}\\ O_{n-2,2} & I_{n-2}\end{pmatrix}.$$

Since $z^{T_2}=-1$ and $Q_1^{T_2}=-I_2$, it follows

$$P(t^1,t^2+T_2)=\begin{pmatrix} z^{-t^2}Q_1^{t^2} & O_{2,n-2}\\ O_{n-2,2} & I_{n-2}\end{pmatrix}=P(t^1,t^2).$$

We verified that $P(t^1+T_1,t^2)=P(t^1,t^2)$, $P(t^1,t^2+T_2)=P(t^1,t^2)$.

Let's check also the conclusion of Theorem 3.

According the Proposition 3, any solution of the recurrence (8) is of the form $y(t^1,t^2)=B_1^{t_1}B_2^{t_2}v$, with $v\in\mathbb{C}^n$. On the other hand, we have

$$P(t^1,t^2)y(t^1,t^2)=Q^{t^2}B_2^{-t^2}B_1^{t_1}B_2^{t_2}v=Q^{t^2}B_1^{t_1}v$$

and we observe that $x(t^1,t^2):=P(t^1,t^2)y(t^1,t^2)$ is indeed a solution of the recurrence (9), which verifies the condition $x(0,0)=v$, according (10).

2.3 Case of periodic coefficients

Definition 3. *Let $T\in\mathbb{N}^m$, $T\neq 0$. The function $f\colon \mathcal{Z}\to M$ is called periodic of period T if $f(t+T)=f(t)$, $\forall t\in\mathcal{Z}$.*

Proposition 7. *Let $T=(T^1,T^2,\dots,T^m)\in\mathbb{N}^m$, $T\neq 0$. We consider the matrix functions $A_\alpha\colon\mathcal{Z}\to\mathcal{M}_n(K)$, $\alpha\in\{1,2,\dots,m\}$, for which the relations (2) are satisfied. Suppose that, $\forall\alpha\in\{1,2,\dots,m\}$, the matrix function $A_\alpha(\cdot)$ is periodic of period T. Let $C(t)=\chi(t+T,t)$. Then*

a) $C_{\alpha,k}(\cdot)$ is periodic of period T;

b) $\chi(t+T,s)=\chi(t,s)\cdot C(s)$, $\forall t,s\in\mathcal{Z}$, with $t\geq s$;

c) $C(s)=C_{1,T^1}(s^1,s^2+T^2,\dots,s^m+T^m)C_{2,T^2}(s^1,s^2,s^3+T^3,\dots,s^m+T^m)\cdot\ldots\cdot$ $\cdot C_{m-1,T^{m-1}}(s^1,s^2,\dots,s^{m-1},s^m+T^m)C_{m,T^m}(s^1,s^2,\dots,s^{m-1},s^m)$, $\forall s\in\mathcal{Z}$.

Proof. *a*) follows from the definition of $C_{\alpha,k}(\cdot)$; this is either the constant function I_n, or a product of matrix functions, periodic with the period T.

b) Fix $s\in\mathcal{Z}$. Let $Y_1,Y_2\colon\{t\in\mathcal{Z}\mid t\geq s\}\to\mathcal{M}_n(K)$,

$$Y_1(t)=\chi(t+T,s),\quad Y_2(t)=\chi(t,s)\cdot C(s),\quad \forall t\geq s.$$

For each $\alpha \in \{1, 2, \ldots, m\}$, we have

$$Y_1(t + 1_\alpha) = \chi(t + T + 1_\alpha, s) = A_\alpha(t + T)\chi(t + T, s) = A_\alpha(t)Y_1(t);$$

$$Y_2(t + 1_\alpha) = \chi(t + 1_\alpha, s) \cdot C(s) = A_\alpha(t)\chi(t, s) \cdot C(s) = A_\alpha(t)Y_2(t).$$

It follows that the functions $Y_1(\cdot)$ and $Y_2(\cdot)$ are both solutions of the recurrence (3); we have also $\chi(s+T, s) = \chi(s, s)C(s)$, i.e., $Y_1(s) = Y_2(s)$. From the uniqueness property (Proposition 1) it follows that $Y_1(\cdot)$ and $Y_2(\cdot)$ coincide; hence $\chi(t + T, s) = \chi(t, s) \cdot C(s)$, $\forall t \geq s$.

c) follows directly from Proposition 2, f). □

Theorem 4. *Let $t_0 \in \mathcal{Z}^m$, fixed. We denote $\mathcal{Z} := \left\{t \in \mathbb{Z}^m \,\middle|\, t \geq t_0\right\}$.*

Let $T = (T^1, T^2, \ldots, T^m) \in \mathbb{N}^m$, $T \neq 0$.

Consider the matrix functions $A_\alpha\colon \mathcal{Z} \to \mathcal{M}_n(\mathcal{C})$, $\alpha \in \{1, 2, \ldots, m\}$, for which the relations (2) are satisfied. Suppose that, $\forall \alpha \in \{1, 2, \ldots, m\}$, the function $A_\alpha(\cdot)$ is periodică de perioadă T and $A_\alpha(t)$ is invertible, $\forall t \in \mathcal{Z}$.

We denote $\Phi(t) := \chi(t, t_0)$, $t \in \mathcal{Z}$.

Then there exists a matrix function $P\colon \mathcal{Z} \to \mathcal{M}_n(\mathcal{C})$, periodic of period T, and also there exist the constant invertible matrix $B \in \mathcal{M}_n(\mathcal{C})$, such that

$$\Phi(t) = P(t)B^{|t|}, \quad \forall t \geq t_0, \text{ where } |t| = t^1 + \ldots + t^m.$$

Proof. The matrices $C_{\alpha, k}(t)$ are invertible (Proposition 2). From the Proposition 7, c), it follows that the matrices $C(s)$ are invertible. Hence there exists an invertible matrix $B \in \mathcal{M}_n(\mathcal{C})$, such that $B^{|T|} = C(t_0)$, where $|T| = T^1 + T^2 + \ldots + T^m$.

We define the function

$$P\colon \mathcal{Z} \to \mathcal{M}_n(\mathcal{C}), \;\; P(t) = \Phi(t)B^{-(|t|}, \quad \forall t \geq t_0.$$

It is sufficient to show that $P(\cdot)$ is periodic of period T.

$$P(t + T) = \chi(t + T, t_0)B^{-|t+T|}.$$

According the Proposition 7, b), we have

$$\chi(t + T, t_0) = \chi(t, t_0)C(t_0) = \Phi(t)B^{|T|}.$$

We obtain

$$P(t + T) = \Phi(t)B^{|T|}B^{-|t+T|} = \Phi(t)B^{-|t|} = P(t).$$

□

Theorem 5. *Let $t_0 \in \mathcal{Z}^m$, fixed. We denote $\mathcal{Z} := \{t \in \mathcal{Z}^m \,|\, t \geq t_0\}$.*

Let $T = (T^1, T^2, \ldots, T^m) \in \mathcal{N}^m$, $T \neq 0$.

Consider the matrix functions $A_\alpha \colon \mathcal{Z} \to \mathcal{M}_n(\mathcal{C})$, $\alpha \in \{1, 2, \ldots, m\}$, for which the relations (2) *are satisfied. Suppose that, $\forall \alpha \in \{1, 2, \ldots, m\}$, $A_\alpha(t)$ is invertible, $\forall t \in \mathcal{Z}$. We denote $\Phi(t) := \chi(t, t_0)$, $t \in \mathcal{Z}$.*

We assume that there exists a matrix function $P \colon \mathcal{Z} \to \mathcal{M}_n(\mathcal{C})$, periodic of period T, and also there exist a constant invertible matrix $B \in \mathcal{M}_n(\mathcal{C})$, such that $\Phi(t) = P(t)B^{|t|}$, $\forall t \geq t_0$.

We consider the recurrences

$$x(t + 1_\alpha) = A_\alpha(t)x(t), \quad \forall t \geq t_0,\ \forall \alpha \in \{1, 2, \ldots, m\}; \tag{11}$$

$$y(t + 1_\alpha) = By(t), \quad \forall t \geq t_0,\ \forall \alpha \in \{1, 2, \ldots, m\}. \tag{12}$$

If $y(t)$ is a solution of the recurrence (12)*, then $x(t) := P(t)y(t)$ is a solution of the recurrence* (11)*. And conversely, if $x(t)$ is a solution of the recurrence* (11)*, then $y(t) := P(t)^{-1}x(t)$ is a solution of the recurrence* (12)*.*

Proof. The matrix $\Phi(t)$ is invertible (Proposition 2).

From $\Phi(t) = P(t)B^{|t|}$ it follows that the matrix $P(t)$ is invertible.

Let $y(t)$ be a solution of the recurrence (12) and $x(t) := P(t)y(t)$; hence $y(t) := P(t)^{-1}x(t)$.

$$y(t + 1_\alpha) = By(t) \Longleftrightarrow P(t + 1_\alpha)^{-1}x(t + 1_\alpha) = BP(t)^{-1}x(t)$$

$$\Longleftrightarrow x(t + 1_\alpha) = P(t + 1_\alpha)BP(t)^{-1}x(t).$$

$$\Longleftrightarrow x(t + 1_\alpha) = \Phi(t + 1_\alpha)B^{-(|t|+1)}BP(t)^{-1}x(t)$$

$$\Longleftrightarrow x(t + 1_\alpha) = A_\alpha(t)\Phi(t)B^{-(|t|)}P(t)^{-1}x(t)$$

$$\Longleftrightarrow x(t + 1_\alpha) = A_\alpha(t)P(t) \cdot P(t)^{-1}x(t) \Longleftrightarrow x(t + 1_\alpha) = A_\alpha(t)x(t).$$

Like it proves the converse. □

3 Discrete multitime Samuelson-Hicks model

We assume that $t = (t^1, ..., t^m) \in \mathbb{N}^m$ is a discrete multitime. Having in mind the discrete single-time Samuelson-Hicks model [10], we introduce a *discrete multitime Samuelson-Hicks like model* based on the following economical elements: (i) two parameters, the first γ, called the *marginal propensity to consume*, subject to $0 < \gamma < 1$, and the second α as *decelerator* if $0 < \alpha < 1$, *keeper* if $\alpha = 1$ or *accelerator* if $\alpha > 1$; (ii) the multiple sequence $Y(t)$ means the *national income* and is the main endogenous variable, the multiple sequence $C(t)$ is the *consumption*; (iii) we assume that multiple sequences $Y(t)$, $C(t)$ are non-negative.

3.1 Constant coefficients Samuelson-Hicks model

We propose a *first order discrete multitime constant coefficients Samuelson-Hicks model* as first order multiple recurrence system

$$\begin{pmatrix} Y(t+1_\beta) \\ C(t+1_\beta) \end{pmatrix} = \begin{pmatrix} \gamma+\alpha & -\frac{\alpha}{\gamma} \\ \gamma & 0 \end{pmatrix} \begin{pmatrix} Y(t) \\ C(t) \end{pmatrix}, \quad \forall t \in \mathbb{N}^m,\ \forall \beta \in \{1,2,\ldots,m\},$$

$$Y(0) = Y_0,\ C(0) = C_0;$$

with $\alpha, \gamma \in \mathbb{C} \setminus \{0\}$.

The matrix of this double recurrence is $A = \begin{pmatrix} \gamma+\alpha & -\frac{\alpha}{\gamma} \\ \gamma & 0 \end{pmatrix}$. According the Example Proposition 3, we have the solution

$$\begin{pmatrix} Y(t) \\ C(t) \end{pmatrix} = A^{|t|} \begin{pmatrix} Y_0 \\ C_0 \end{pmatrix}. \tag{13}$$

Let r_1, r_2 be the roots of the characteristic polynomial $r^2 - (\gamma+\alpha)r + \alpha$ of the matrix A. It proves easily by induction that whether $r_1 \neq r_2$, then

$$A^k = \frac{r_2^k - r_1^k}{r_2 - r_1} \cdot A + \frac{r_2 r_1^k - r_1 r_2^k}{r_2 - r_1} \cdot I_2, \quad \forall k \in \mathbb{N}, \tag{14}$$

and whether $r_1 = r_2$, then

$$A^k = k r_1^{k-1} A - (k-1) r_1^k I_2, \quad \forall k \in \mathbb{N}. \tag{15}$$

We set $k = |t| = t^1 + t^2 + \ldots + t^m$ in the formula (14) or (15) and we obtain the relation (13) which gives concrete expressions for $Y(t)$, $C(t)$ in function of r_1, r_2.

3.2 Multi-periodic coefficients Samuelson-Hicks model

Let us use the variable parameters

$$\alpha\colon \mathbb{N}^m \to \mathbb{C},\ \gamma\colon \bigcup_{\beta=1}^{m} \left\{ t \in \mathbb{Z}^m \,\middle|\, t \geq -1_\beta \right\} \to \mathbb{C},$$

such that $\gamma(t) \neq 0$ and $\gamma(t) + \alpha(t) \notin \{0; 1\}$, $\forall t \in \mathbb{N}^m$. They define a *discrete multitime multiple Samuelson-Hicks model*

$$\begin{pmatrix} Y(t+1_\beta) \\ C(t+1_\beta) \end{pmatrix} = A_\beta(t) \begin{pmatrix} Y(t) \\ C(t) \end{pmatrix}, \quad \forall t \in \mathbb{N}^m,\ \forall \beta \in \{1,2,\ldots,m\},$$

$$Y(0) = Y_0,\ C(0) = C_0.$$

The matrix

$$A_\beta(t) = \begin{pmatrix} \gamma(t) + \alpha(t) & -\frac{\alpha(t)}{\gamma(t-1_\beta)} \\ \gamma(t) & 0 \end{pmatrix}, \quad \forall t \in \mathbb{N}^m,\ \forall \beta \in \{1, 2, \ldots, m\}, \tag{16}$$

must satisfy the relations (2), i.e.

$$A_\beta(t + 1_\mu) A_\mu(t) = A_\mu(t + 1_\beta) A_\beta(t), \tag{17}$$

$$\forall t \in \mathbb{N}^m,\ \forall \beta, \mu \in \{1, 2, \ldots, m\}.$$

We denote $x(t) = \begin{pmatrix} Y(t) \\ C(t) \end{pmatrix}$. The Samuelson-Hicks recurrence writes

$$x(t + 1_\beta) = A_\beta(t) x(t), \quad \forall t \in \mathbb{N}^m,\ \forall \beta \in \{1, 2, \ldots, m\}, \tag{18}$$

$$x(0) = x_0.$$

The relation (17) is equivalent to

$$\begin{pmatrix} \big(\gamma(t) + \alpha(t)\big)\big(\gamma(t + 1_\mu) + \alpha(t + 1_\mu)\big) - \frac{\gamma(t)\alpha(t+1_\mu)}{\gamma(t+1_\mu-1_\beta)} & * \\ \big(\gamma(t) + \alpha(t)\big)\gamma(t + 1_\mu) & * \end{pmatrix}$$

$$= \begin{pmatrix} \big(\gamma(t) + \alpha(t)\big)\big(\gamma(t + 1_\beta) + \alpha(t + 1_\beta)\big) - \frac{\gamma(t)\alpha(t+1_\beta)}{\gamma(t+1_\beta-1_\mu)} & * \\ \big(\gamma(t) + \alpha(t)\big)\gamma(t + 1_\beta) & * \end{pmatrix}.$$

It follows that

$$\gamma(t + 1_\mu) = \gamma(t + 1_\beta)$$

and

$$\begin{aligned} &\big(\gamma(t) + \alpha(t)\big)\alpha(t + 1_\mu) - \frac{\gamma(t)\alpha(t + 1_\mu)}{\gamma(t + 1_\mu - 1_\beta)} \\ &= \big(\gamma(t) + \alpha(t)\big)\alpha(t + 1_\beta) - \frac{\gamma(t)\alpha(t + 1_\beta)}{\gamma(t + 1_\beta - 1_\mu)}. \end{aligned} \tag{19}$$

By induction, one obtains $\gamma(t + k \cdot 1_\mu) = \gamma(t + k \cdot 1_\beta)$, $\forall k \in \mathbb{N}$, $\forall t \in \mathbb{N}^m$, $\forall \mu$, $\forall \beta$ and

$$\gamma(t) = \gamma\big((t^1, t^2, \ldots, t^{m-1}, 0) + t^m \cdot 1_m\big) = \gamma\big((t^1, t^2, \ldots, t^{m-1}, 0) + t^m \cdot 1_1\big)$$

$$= \gamma(t^1 + t^m, t^2, \ldots, t^{m-1}, 0) = \gamma\big((t^1 + t^m, t^2, \ldots, t^{m-2}, 0, 0) + t^{m-1} \cdot 1_{m-1}\big)$$

$$= \gamma\big((t^1 + t^m, t^2, \ldots, t^{m-2}, 0, 0) + t^{m-1} \cdot 1_1\big) = \gamma(t^1 + t^{m-1} + t^m, t^2, \ldots, t^{m-2}, 0, 0)$$

$$= \ldots = \gamma(|t|, 0, \ldots, 0).$$

where $|t|$ means the sum $t^1 + ... + t^m$.

Let $f(k) := \gamma(k, 0, \ldots, 0)$, $k \in \mathbb{N} \cup \{-1\}$; we have obtained $\gamma(t) = f(|t|)$. The relation (19) becomes

$$\big(\gamma(t) + \alpha(t)\big)\alpha(t + 1_\mu) - \alpha(t + 1_\mu) = \big(\gamma(t) + \alpha(t)\big)\alpha(t + 1_\beta) - \alpha(t + 1_\beta)$$

if and only if $\gamma(t) + \alpha(t) \neq 1$. Then $\alpha(t + 1_\mu) = \alpha(t + 1_\beta)$; analogously to $\gamma(\cdot)$, it is shown that there exists $g(k)$, $k \in \mathbb{N}$, such that $\alpha(t) = g(|t|)$.

We denote $A(k) = \begin{pmatrix} f(k) + g(k) & -\dfrac{g(k)}{f(k-1)} \\ f(k) & 0 \end{pmatrix}$, $k \in \mathbb{N}$. On the other hand, we have $A_\beta(t) = A(|t|)$, $\forall t \in \mathbb{N}^m$, $\forall \beta \in \{1, 2, \ldots, m\}$; and immediately notice now that, in this situation, the relationships (17) are satisfied.

We showed that *the relations* (17) *are satisfied if and only if there exist the functions* $f\colon \mathbb{N} \cup \{-1\} \to \mathbb{C}$, $g\colon \mathbb{N} \to \mathbb{C}$, *such that*

$$\gamma(t) = f(|t|), \quad \forall t \in \bigcup_{\beta=1}^{m} \big\{ t \in \mathbb{Z}^m \,\big|\, t \geq -1_\beta \big\}; \;\; \alpha(t) = g(|t|), \quad \forall t \in \mathbb{N}^m.$$

We consider the recurrence

$$z(k+1) = A(k) z(k), \quad \forall k \in \mathbb{N}. \tag{20}$$

Let $z\colon \mathbb{N} \to \mathbb{C}^2$ be the solution of the recurrence (20), with $z(0) = x_0$. Let $x\colon \mathbb{N}^m \to \mathbb{C}^2$, $x(t) = z(|t|)$, $\forall t \in \mathbb{N}^m$. Then

$$x(t + 1_\beta) = z(|t| + 1) = A(|t|) z(|t|) = A_\beta(t) x(t).$$

Obviously $x(0, 0, \ldots, 0) = z(0) = x_0$. Hence, the following result is true.

If $z\colon \mathbb{N} \to \mathbb{C}^2$ *is the solution of the recurrence* (20), *with* $z(0) = x_0$, *then the function*

$$x\colon \mathbb{N}^m \to \mathbb{C}^2, \;\; x(t) = z(|t|), \quad \forall t \in \mathbb{N}^m$$

is the solution of the recurrence (18), *which verifies* $x(0) = x_0$.

Let $p \in \mathbb{N}$ and

$$C_p\colon \mathbb{N} \to \mathcal{M}_2(\mathbb{C}),$$

$$C_p(k) = \begin{cases} \displaystyle\prod_{\ell=1}^{p} A(k + p - \ell), & \text{if } p \geq 1 \\ I_2, & \text{if } p = 0. \end{cases}$$

The solution of the recurrence (20), with $z(0) = x_0$ is $z(k) = C_k(0)x_0$, $\forall k \in \mathbb{N}$. Hence, the solution of the recurrence (18), which verifies $x(0) = x_0$, is

$$x(t) = C_{|t|}(0)x_0, \ \forall t \in \mathbb{N}^m.$$

Let $\chi(\cdot, \cdot)$ the fundamental matrix associated to the recurrence (18) and let $\chi_1(\cdot, \cdot)$ the fundamental matrix associated to the recurrence (20). We denote $\Phi(t) = \chi(t, 0)$, $t \in \mathbb{N}^m$, and $\Psi(k) = \chi_1(k, 0)$, $k \in \mathbb{N}$.

From the above observations immediately yield $\Psi(k) = C_k(0)$, $\forall k \in \mathbb{N}$, and $\Phi(t) = C_{|t|}(0)$, $\forall t \in \mathbb{N}^m$. Hence $\Phi(t) = \Psi(|t|)$, $\forall t \in \mathbb{N}^m$.

The foregoing first order homogeneous multiple recurrence is called *multi-periodic* if its coefficients are multi-periodic or if the matrix $A_\mu(t)$ is multi-periodic. A prominent role in the analysis of a multi-periodic recurrence is played by so-called *Floquet multipliers.*

Let us consider a *discrete multitime multi-periodic coefficients Samuelson-Hicks model.* For that, suppose $\exists \mu$, $\exists \delta$ and $\exists T \in \mathbb{N}^*$ such that $A_\mu(t+T\cdot 1_\delta) = A_\mu(t)$, $\forall t \in \mathbb{N}^m$; equivalent to $A(t^1+t^2+\ldots+t^m+T) = A(t^1+t^2+\ldots+t^m)$, $\forall t \in \mathbb{N}^m$. Set $t^1 = k$ and $t^\beta = 0$, for $\beta \geq 2$. One obtains $A(k+T) = A(k)$, $\forall k \in \mathbb{N}$. And here follows immediately that $A_\beta(t+T\cdot 1_\rho) = A_\beta(t)$, $\forall t \in \mathbb{N}^m$, $\forall \rho$, $\forall \beta$.

Consequently,

if $\exists \mu$, $\exists \delta$ and $\exists T \in \mathbb{N}^*$ a.î. $A_\mu(t+T\cdot 1_\delta) = A_\mu(t)$, $\forall t \in \mathbb{N}^m$, then

$A_\beta(t+T\cdot 1_\rho) = A_\beta(t)$, $\forall t \in \mathbb{N}^m$, $\forall \rho$, $\forall \beta$

and the function $A(\cdot)$ is in fact periodic with the period T; which is equivalent to the fact that the functions f and g are periodic, of multi-period T.

Let us suppose that the matrices $A(k)$ are invertible; it appears the condition $g(k) \neq 0$, $\forall k \in \mathbb{N}$.

Let us compute the matrices $\widetilde{C}_\beta = C_{\beta,T_\beta}(t_0)$, for the recurrences (18), (20), with $t_0 = 0$. Obviously, in the case of the recurrence (20), we have $T_\beta = T$, $\forall \beta$.

For (18), we have a single such matrix, namely $\widetilde{C} = C_T(0) = \Psi(0)$.

For the recurrence (20), we have $\widetilde{C}_\beta = C_{\beta,T}(0) = C_T(0) = \widetilde{C}$.

If the matrices $A(k)$ are invertible, then $\widetilde{C}$ is invertible; hence, there exists $B \in \mathcal{M}_2(\mathbb{C})$, such that $B^T = \widetilde{C}$. Obviously, this is equivalent to $B^{T_\beta} = \widetilde{C}_\beta$ (hence $B_\beta = B$, $\forall \beta$).

Hence, we can apply the Proposition 6 for the recurrence (18) (is in fact the case $m = 1$) and (20) (the matrices B_β commute, since they are equal).

Consequently, there exists $R(k)$ and $P(t)$ such that

$$R(k+T) = R(k), \quad \forall k \in \mathbb{N},$$
$$\Psi(k) = R(k)B^k, \quad \forall k \in \mathbb{N},$$
$$P(t + T \cdot 1_\beta) = P(t), \quad \forall t \in \mathbb{N}^m, \ \forall \beta,$$
$$\Phi(t) = P(t)B^{|t|}, \quad \forall t \in \mathbb{N}^m.$$

But $\Phi(t) = \Psi(|t|) \Longleftrightarrow P(t)B^{|t|} = R(|t|)B^{|t|} \Longleftrightarrow P(t) = R(|t|)$. Hence, we have obtained: $P(t) = R(|t|)$ and $\Phi(t) = R(|t|)B^{|t|}$, $\forall t \in \mathbb{N}^m$.

Suppose that we are in case of multi-periodic recurrence of the type (18) (discrete multitime multi-periodic coefficients Samuelson-Hicks), i.e., the functions f are g periodic, with the period $T \geq 1$ (equivalent to $A(\cdot)$ is periodic with the period T).

The matrix

$$\widetilde{C} = \widetilde{C}_\beta = C_{\beta, T_\beta}(0) = C_{\beta, T}(0) = C_T(0) = \prod_{j=1}^{T} A(T-j)$$

is called *monodromy matrix* associated to the (multi-periodic) recurrence (18).

According the Proposition 4, c), we have $\Psi(k + T \cdot 1_\beta) = \Psi(k) \cdot \widetilde{C}$, $\forall k \in \mathbb{N}$. By induction, it follows $\Psi(k + pT \cdot 1_\beta) = \Psi(k) \cdot (\widetilde{C})^p$, $\forall p \in \mathbb{N}$, $\forall k \in \mathbb{N}$.

The Floquet multipliers of the multi-periodic recurrence (18), *are the two roots of the quadratic equation*

$$\lambda^2 - (Tr\ \widetilde{C})\lambda + \det \widetilde{C} = 0.$$

It is easy to see that $\det \widetilde{C} = \dfrac{f(T-1)}{f(-1)} \displaystyle\prod_{j=0}^{T-1} g(j)$.

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